Froissart-Stora Revisited

1 Equation of motion for arbitrary resonance crossing pattern

Assume one and only one depolarization resonance of strength $\epsilon$ at $a\gamma = \kappa$.

\[
\frac{d\psi}{d\theta} = -i \frac{1}{2} H \psi
\]

\[
H = \begin{bmatrix}
-a\gamma & \epsilon e^{i\kappa \theta} \\
\epsilon^* e^{-i\kappa \theta} & a\gamma
\end{bmatrix}
\]

\[
a\gamma(\theta) = \kappa + \alpha(\theta)
\]

\[
\alpha(\theta) = \text{arbitrary}
\]

Make the transformation

\[
\psi_1 = e^{-\frac{i}{2} [\kappa \theta + \beta(\theta)]} \sigma_y \psi
\]

\[
\beta(\theta) = \int_{-\infty}^{\theta} d\theta' \alpha(\theta')
\]

\[
\Rightarrow \quad \frac{d\psi_1}{d\theta} = -i \frac{1}{2} \begin{bmatrix}
0 & \epsilon e^{-i\beta(\theta)} \\
\epsilon^* e^{i\beta(\theta)} & 0
\end{bmatrix} \psi_1
\]

We now let

\[
\psi_1(\theta) = \begin{bmatrix} f(\theta) \\ g(\theta) \end{bmatrix} \quad \Rightarrow \quad \begin{cases}
|f|^2 + |g|^2 = 1 \\
f''(\theta) + i\alpha(\theta)f'(\theta) + \frac{|\epsilon|^2}{4}f(\theta) = 0
\end{cases}
\]

Polarization is

\[
P(\theta) = \psi_1^\dagger \sigma_y \psi_1 = \psi_1^\dagger \sigma_y \psi_1
\]

\[
= |f(\theta)|^2 - |g(\theta)|^2 = 2|f(\theta)|^2 - 1
\]

Our job is to solve for $f(\theta)$ given $\alpha(\theta)$. 1
2 Initial condition

Initial condition is specified at a large negative $\theta_{\text{min}}$. Let $\alpha_{\text{min}} = \alpha(\theta_{\text{min}})$. Assume the spin has been in an eigenstate from $\theta = -\infty$ till $\theta = \theta_{\text{min}}$:

\[
\begin{align*}
f(\theta_{\text{min}}) &= \sqrt{\frac{\sqrt{\alpha_{\text{min}}^2 + |\epsilon|^2 + |\alpha_{\text{min}}|^2}}{2\sqrt{\alpha_{\text{min}}^2 + |\epsilon|^2}}} \\
g(\theta_{\text{min}}) &= -\frac{|\epsilon|}{\epsilon} \text{sgn}(\alpha_{\text{min}}) \sqrt{\frac{\sqrt{\alpha_{\text{min}}^2 + |\epsilon|^2 - |\alpha_{\text{min}}|^2}}{2\sqrt{\alpha_{\text{min}}^2 + |\epsilon|^2}}}
\end{align*}
\]

The initial polarization is not 100%,

\[P(\theta_{\text{min}}) = |f(\theta_{\text{min}})|^2 - |g(\theta_{\text{min}})|^2 = \frac{|\alpha_{\text{min}}|}{\sqrt{\alpha_{\text{min}}^2 + |\epsilon|^2}}\]

If spin tune continues to be $a\gamma = \kappa + \alpha_{\text{min}}$ after $\theta_{\text{min}}$,

\[
\begin{align*}
f(\theta > \theta_{\text{min}}) &= f(\theta_{\text{min}}) \exp \left[ \frac{i(\theta - \theta_{\text{min}})}{2} \text{sgn}(\alpha_{\text{min}}) \left( \sqrt{\alpha_{\text{min}}^2 + |\epsilon|^2 - |\alpha_{\text{min}}|^2} \right) \right] \\
g(\theta > \theta_{\text{min}}) &= g(\theta_{\text{min}}) \exp \left[ \frac{i(\theta - \theta_{\text{min}})}{2} \text{sgn}(\alpha_{\text{min}}) \left( \sqrt{\alpha_{\text{min}}^2 + |\epsilon|^2 + |\alpha_{\text{min}}|^2} \right) \right]
\end{align*}
\]

A numerical Mathematica code is written to solve $f$ with these initial conditions.
3 Linear crossing – Froissart-Stora case

Froissart-Stora solved analytically

$$\alpha_{FS}(\theta) = \alpha_0 \theta$$

Numerical calculation for the Froissart-Stora case with \( \alpha_0 = 0.01 \) and \(|\epsilon| = 0.1\). \( P(\theta \to \infty) \) is to be compared with Froissart-Stora formula \( P_{FS}(\infty) = 2e^{-\pi|\epsilon|^2/2\alpha_0} - 1 = -0.584 \).
4 Step crossing

\[ \alpha(\theta) = \begin{cases} A, & \text{if } \theta > 0 \\ -A, & \text{if } \theta < 0 \end{cases} \]

Result for \(|\epsilon| = 0.1\) and \(A = 0.1 \pi\):

But this case can be solve analytically also. For \(\theta < 0\), the spin is in the eigenstate. For \(\theta > 0\), the polarization precesses,

\[
P(\theta) = \left( 1 + \frac{|A|}{2\Omega} \right) \left( \cos^2 \Omega \theta + \frac{(|A| - \Omega)^2}{\Omega^2} \sin^2 \Omega \theta \right) - 1
\]
\[ \Omega = \frac{\sqrt{A^2 + |\epsilon|^2}}{2} \]

\[ \langle P \rangle = \frac{|A|(A^2 - |\epsilon|^2)}{(A^2 + |\epsilon|^2)^{3/2}} \] (averaged over time)

In comparison, the Froissart-Stora formula yields a final polarization of 100%. Our numerical example (which may be considered to satisfy \( A \gg |\epsilon| \)) can give up to 40% discrepancy!
5 Tapered-step crossing

$$\alpha(\theta) = A \tan^{-1}\left(\frac{\alpha_0 \theta}{A}\right)$$

where $\alpha_0$ is the linear crossing speed at the moment $\theta = 0$. The spin tune swings from $-\frac{\pi A}{2}$ to $\frac{\pi A}{2}$.

Result for $\alpha_0 = 0.01$, $|\epsilon| = 0.1$, and $A = 0.2$. Froissart-Stora predicts a final polarization of -0.584. Agreement is better than step-crossing but note the large final state oscillation.
6 Sloped-step crossing

\[ \alpha(\theta) = \begin{cases} 
\alpha_0 \theta, & \text{if } \frac{A}{\alpha_0} > \theta > -\frac{A}{\alpha_0} \\
A, & \text{if } \theta > \frac{A}{\alpha_0} \\
-A, & \text{if } \theta < -\frac{A}{\alpha_0}
\end{cases} \]

The spin tune swings from \(-A\) to \(A\) linearly.

Result for \(\alpha_0 = 0.01\), \(|\epsilon| = 0.1\), and \(A = 0.1 \pi\). Froissart-Stora predicts -0.584.
7 Sloped-step crossing – analytic solution

But the sloped-step case can be solved analytically!

Make transformation

\[ F(\theta) = f(\theta)e^{\frac{1}{2}\int_{\theta_{\text{min}}}^{\theta} d\theta' \alpha(\theta')} \]

\[ \implies \begin{cases} F'' + \left( \frac{\alpha^2 + |\epsilon|^2}{4} - i\frac{\alpha'}{2} \right) F = 0 \\ P(\theta) = 2|F(\theta)|^2 - 1 \end{cases} \]

Duration of linear crossing when \( \alpha = \alpha_0 \), make another transformation \( x = \sqrt{\alpha_0} \theta \), then

\[ \frac{d^2 F}{dx^2} + \left( \frac{x^2}{4} - a \right) F = 0 \]

\[ a = \frac{i}{2} - \frac{|\epsilon|^2}{4\alpha_0} \]

Two independent solutions for \( F \) can be expressed in terms of hypergeometric function \( _1F_1 \),

\[ y_1(x) = e^{-i\frac{x^2}{4}} x^{rac{3}{4} + i\frac{a}{2}} \left( \frac{3}{4} + i\frac{a}{2} - i\frac{x^2}{2} \right) \]

\[ y_2(x) = xe^{i\frac{x^2}{4}} x^{-\frac{1}{4} - i\frac{a}{2}} \left( 1 - i\frac{a}{2}; \frac{1}{2} \right) \]

General solution is

\[ F(x) = C_1 y_1(x) + C_2 y_2(x) \]

Initial condition at time \( \theta_{\text{min}} \) is the eigenstate, which yields

\[ C_1 = \left( Y_2' - \frac{ir}{2\sqrt{\alpha_0}} Y_2 \right) \cos \phi \]

\[ C_2 = \left( Y_1' - \frac{ir}{2\sqrt{\alpha_0}} Y_1 \right) \cos \phi \]
where

\[ A = r \cos 2\phi, \quad |\epsilon| = r \sin 2\phi, \quad Y_k = y_k \left( \frac{A}{\sqrt{\alpha_0}} \right), \quad Y'_k = y'_k \left( \frac{A}{\sqrt{\alpha_0}} \right) \]

Polarization is

\[
P(\theta) = \begin{cases} 
\cos 2\phi, & \text{if } \theta < \theta_{\min} = -\frac{A}{\alpha_0} \\
2 \cos^2 \phi \left| y_1 Y'_2 + y_2 Y'_1 - \frac{ir}{2\sqrt{\alpha_0}} (y_1 Y_2 + y_2 Y_1) \right|^2 - 1 , & \text{from } \theta = \theta_{\min} \text{ to } \theta = \theta_{\max} \\
y_{1,2} \text{ are evaluated at } \sqrt{\alpha_0} \theta \\
2 \left| \sqrt{\alpha_0} \Omega (C_1 Y'_1 + C_2 Y'_2) \sin \Theta + (C_1 Y_1 + C_2 Y_2) \cos \Theta \right|^2 - 1 , & \text{if } \theta > \theta_{\max} = \frac{A}{\alpha_0} \\
\Theta = \Omega (\theta - \theta_{\max}), & \Omega = \sqrt{\frac{A^2 + |\epsilon|^2}{2}}
\end{cases}
\]

Results for three cases:

- \( A/\sqrt{\alpha_0} = \pi, \quad |\epsilon|/\sqrt{\alpha_0} = 1 \) red
- \( A/\sqrt{\alpha_0} = \pi, \quad |\epsilon|/\sqrt{\alpha_0} = 4 \) green
- \( A/\sqrt{\alpha_0} = 1, \quad |\epsilon|/\sqrt{\alpha_0} = 1 \) blue
What is the “final polarization”?
For $\theta > \theta_{\text{max}}$, the polarization oscillates sinusoidally with frequency $2\Omega = \sqrt{A^2 + |\epsilon|^2}$,

\[
P = (|D_1|^2 + |D_2|^2 - 1) \pm |D_1^2 + D_2^2| \\
D_1 = \frac{1}{\sqrt{A^2 + \epsilon^2}} (C_1Y_1' + C_2Y_2') \\
D_2 = C_1Y_1 + C_2Y_2
\]
Four figures correspond to \( \tilde{A} = \frac{A}{2\sqrt{\alpha_0}} = 0.5, 1, 2, 10 \). Horizontal axes are \( \tilde{\epsilon} = \frac{|\epsilon|}{2\sqrt{\alpha_0}} \). Vertical axes give final polarization after resonance crossing. Green curves give Froissart-Stora result \( P_{\text{final}} = 2e^{-2\pi\tilde{\epsilon}^2} - 1 \).

The above figures can be applied as follows:

- From value of \( \tilde{A} = \frac{A}{2\sqrt{\alpha_0}} \) in the experiment, determine which figure is to be used.
- Determine the range of \( \tilde{\epsilon} = \frac{|\epsilon|}{2\sqrt{\alpha_0}} \) consistent with the measured final polarization.
- Compare the range with the Froissart-Stora prediction.

For example, if \( \tilde{A} = 0.5 \) and \( P = -50\% \), the range of \( \tilde{\epsilon} \) is found to be from 0.37 to 1.27, while the Froissart-Stora prediction would give \( \tilde{\epsilon} = 0.47 \).
The Generalized Froissart-Stora contains rich experimental information, e.g.

- Polarization after crossing differs from Froissart-Stora when $A \lesssim |\epsilon|$

- Beating frequency after crossing offers another way to measure $|\epsilon|$. Can one do an FFT to extract the beating frequency?

- What does it mean when solution for $|\epsilon|$ is double valued for a given measured final polarization? Could this contribute to the very large discrepancy in $\epsilon_{\text{eff}}/\epsilon$??
8 Effect of error bars

After each measurement of $P(\theta, \alpha_0, A, |\epsilon|)$, we use Froissart-Stora formula to calculate an effective $|\epsilon|_{\text{eff}}$,

$$|\epsilon|_{\text{eff}} = \sqrt{\frac{2\alpha_0}{\pi} \ln \left( \frac{2}{P(\theta, \alpha_0, A, |\epsilon|) + 1} \right)}$$

How does measurement errors affect $|\epsilon|_{\text{eff}}$? Assume the Froissart-Stora formula is exact but now there is an error in the measurements.

$|\epsilon|_{\text{eff}}/|\epsilon|$ as a function of $|\epsilon|/\sqrt{\alpha_0}$. Red line = no error. Two green curves are when measurement is 1% and 5% higher than the exact formula. Two blue curves are when it is 1% and 5% lower.

For an accuracy $\sim 10-20\%$ in $|\epsilon|_{\text{eff}}$, should stay within $|\epsilon|/\sqrt{\alpha_0} \sim 0.2$ to 1.5.
9 Adding synchrotron oscillation

\[ \alpha(\theta) = \begin{cases} 
\alpha_0 \theta, & \text{if } \frac{A}{\alpha_0} > \theta > -\frac{A}{\alpha_0} \\
A, & \text{if } \theta > \frac{A}{\alpha_0} \\
-A, & \text{if } \theta < -\frac{A}{\alpha_0} 
\end{cases} + A_m \cos(\nu_m \theta + \phi_m) \]

We numerically calculate the time-averaged polarization \( \langle P \rangle \) and extract \( |\epsilon|_{\text{eff}} \). Result of \( |\epsilon|_{\text{eff}} \) versus \( A_m \) for the case \( \alpha_0 = 0.01, A = 0.1 \pi, |\epsilon| = 0.1 \). Red, green, blue curves are for the cases \((\nu_m = 0.1, \phi_m = 0), (\nu_m = 0.1, \phi_m = \pi/2), (\nu_m = 0.2, \phi_m = 0)\). Substantial deviation from the Froissart-Stora formula is seen when \( |A_m| > 0.05 \).
10 Two nearby resonances

How far apart the two resonances have to be in order to apply the Froissart-Stora formula?

\[
\psi_1 = e^{-i \frac{1}{2} \int_{-\infty}^{\theta} d\theta' a_{\gamma}(\theta')} \psi_y
\]

\[
\frac{d\psi_1}{d\theta} = -\frac{i}{2} \begin{bmatrix}
\varepsilon e^{i \int_{-\infty}^{\theta} d\theta' a_{\gamma}(\theta')} & \varepsilon e^{-i \int_{-\infty}^{\theta} d\theta' a_{\gamma}(\theta')} \\
0 & 0
\end{bmatrix} \psi_1
\]

\[\varepsilon = \epsilon_1 e^{i \kappa_1 \theta} + \epsilon_2 e^{i \kappa_2 \theta}\]

\[a_{\gamma}(\theta) = \kappa_0 + \alpha_0 \theta\]

\[\Delta \kappa_1 = \kappa_1 - \kappa_0\]

\[\Delta \kappa_2 = \kappa_2 - \kappa_0\]

Numerical example: \(\epsilon_1 = 0.0015+0.0015 i, \epsilon_2 = 0.002, \alpha_0 = 0.00005, \Delta \kappa_1 = 0.05, \Delta \kappa_2 = -0.05\). Final polarization is to be compared with \((2e^{-\pi |\epsilon_1|^2/2\alpha_0} - 1)(2e^{-\pi |\epsilon_2|^2/2\alpha_0} - 1) = 0.562\).

The two resonances are “far apart” with \(|\Delta \kappa| \gg |\epsilon|\), but interference remains very pronounced!

Red curve = case above. Green/blue curve = when second resonance is slightly shifted to \(\Delta \kappa_2 = -0.051/ - 0.052\).
11 Can two nearby resonances be made to cancel each other?

The blue curve hints this possibility. If true, one may introduce an artificial resonance (rf dipole/solenoid) to cancel another unavoidable resonance (intrinsic, or maybe partially imperfection resonance).

“Spin precession bump”: spin motion is affected during the time between the two crossings, but little effect outside.

To cancel, want

\[ |\epsilon_1| = |\epsilon_2| \]

adjust resonance spacing \( \kappa_2 - \kappa_1 \) to a right value

The first condition by choosing the strength of the rf dipole. The second condition by adjusting the rf frequency.

“Effective resonance strength” found semi-empirically,

\[ |\epsilon_{\text{EFF}}|^2 = 2|\epsilon|^2 \left[ 1 + \cos \left( \phi_1 - \phi_2 + \frac{(\kappa_2 - \kappa_1)^2}{3.97 \alpha_0} \right) \right] \]

where \( \epsilon_{1,2} = |\epsilon|e^{i\phi_{1,2}} \). \( |\epsilon_{\text{EFF}}| \) varies between 0 (destructive interference) and 2|\epsilon_1| (constructive interference).

Cancellation should be periodic

\[ \kappa_{\text{period}} = \frac{3.97 \pi \alpha_0}{|\kappa_2 - \kappa_1|} \]
Example: Final polarization as a function of $\kappa_1 - \kappa_2$ with $\epsilon_1 = 0.002i, \epsilon_2 = 0.002, \alpha_0 = 0.00005, \Delta\kappa_1 = 0.05$. Green curve uses the effective resonance strength.

Single-particle spin motion remembers past resonance crossings forever. However, for a beam of particles, the memory lasts a maximum resonance separation of 

$$|\kappa_2 - \kappa_1|_{\text{max}} \sim \frac{\alpha_0}{a\sigma_\gamma} ?$$

Maybe useful to design experiments to study

- cancellation of a intrinsic resonance by an rf dipole resonance
- double/repeated crossing of a single resonance – like in the spin flip experiments
- crossing two split half-integer resonances of a 99% snake – suggested by Morozov
- crossing $a\gamma = n \pm Q_\beta$ resonances when $Q_\beta$ is sufficiently close to integer – suggested by Courant
12 Summary

Analytic solution:

- spin tune = constant, no crossing
- linear crossing (Froissart-Stora)
- step crossing
- sloped-step crossing, generalized Froissart-Stora

Numerical solution by Mathematica code:

- checked the soluble cases
- other crossing patterns
- synchrotron oscillation
- two nearby resonances

Other effects:

- error bars
- two far resonances can be made to cancel each other
- possibility of new experiments, sloped-step crossing
- possibility of new experiments, double crossing